Available online at www.sciencedirect.com
ScienceDirect
International Journal of Heat and Mass Transfer 51 (2008) 865–872
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Temperature distribution in a rectangular plate heated by a moving heat source

J. Kidawa-Kukla

Institute of Mechanics and Machine Design, Częstochowa University of Technology, Dąbrowskiego 73, 42-200 Częstochowa, Poland

Available online 29 October 2007

Abstract

In this paper, a solution to the problem of heat conduction in a rectangular plate subjected to the activity of a moving heat source is presented. The temperature of the plate changes because a limited area on the plate surface is heated by a heat source. The heat source moves along an elliptical trajectory which always remains within the plate area. An exact solution to the problem in an analytical form is obtained by applying the Green's function method. Exemplary results of numerical calculations to determine the temperature distribution in the plate are presented.

Keywords: Heat conduction; Moving heat source; Temperature distribution in a plate; Green's function method

1. Introduction

Mathematical models of heat conduction with moving heat sources have practical applications in numerous manufacturing processes such as welding, metal cutting, and the grinding and drilling of metals. The problem is the subject of many publications (for instance, Refs. [1–6]), and authors often use analytical methods to solve these types of problems. The application of analytical models of heat conduction is limited in regular domains, but their use is more profitable in the analysis of the process.

An analytical solution to the problem of the three-dimensional temperature distribution generating by a moving laser beam source in a finite domain was presented by Araya and Gutierrez [1]. The heat source is modeled as a laser beam with a Gaussian distribution or as a spatially uniform plane. The results and discussion concern the boundary effects on the temperature of a workpiece. An analytical model which describes a three-dimensional temperature fields in a finite thickness plate was investigated by Cheng and Lin [2]. The considered plate was heated by a moving heat source with a Gaussian distribution. It was assumed that the heat source moved with a constant velocity along a line which was parallel to one edge of the rectangular plate. A circular Gaussian heat source moving at a constant relative velocity over the surface of a solid with a finite depth and width was also assumed by Manca et al. [3]. An analytical form of the solution to the three-dimensional problem was derived using a Green's function method (GFM). The Green's function describes the temperature distribution caused by an instantaneous, local energy pulse. The specific use of the GFM in solving various heat conduction problems was widely discussed by Beck et al. [4]. Examples of applications of the GFM in solving heat conduction problems in beams and a plate with moving heat sources are presented by Kidawa-Kukla [5–7]. In Refs. [5,6] analytical solutions to the temperature distribution in rectangular beams heated by a moving heat source which moves harmonically around a fixed point on the beam surface are presented. These solutions were then used to determine the displacement of the beam induced by cyclic changes in the temperature. A parabolic equation [5] or a hyperbolic equation [6] of the heat conduction was assumed in the mathematical model. The temperature distribution in a rectangular plate subjected to the activity of a

E-mail address: jk@imipkm.pcz.czest.pl

0017-9310/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2007.04.011

Nomenclature

$T(x,y,z,t)$	temperature of a plate (°C)
$g(x,y,t)$	volumetric energy of heat source (W m^{-3})
$\bar{x}(t), \bar{y}(t)$	functions describing the movement of the heat source (m)
T_0, T_1	temperatures of a surrounding medium (°C)
a, b, h	dimensions of a plate (m)
t	time (s)
x_0, y_0	coordinates of fixed point of a plate (m)
$G(x,y,z,t,\xi,\eta,\varsigma,\tau)$	Green's function

Greek symbols

κ	thermal diffusivity (m^2 s^{-1})
Θ	heat flux of source (W m^{-2})
ε	size of the quadratic element on the plate surface heated by the heat source (m)
$\delta()$	Dirac delta function
α_0, α_1	heat transfer coefficients (W m^{-1} K^{-1})
λ	thermal conductivity (W m^{-1} K^{-1})
φ	angular velocity of the moving heat source (rad s^{-1})

Subscripts

i, j, k, l, m, n indices

heat source is presented in Ref. [7]. It was assumed that the heat source moves harmonically around a fixed point along a segment which always remains on the plate.

The analytical form of the solution to the non-homogeneous heat conduction problem usually involves an infinite series which is characterized by slow convergence. For example, low convergence can be observed in the steady state part of solutions to the partial heating of solids. The verification of solution methods to the heat transfer problem for the partial heating of a rectangular solid is the subject of the paper by Beck et al. [8]. In order to improve the convergence of the solution, the authors of papers [4,8] recommend the use of the GFM in conjunction with a time-partition method. In this approach the large and the small-time forms of the Green's functions are applied.

The first step in solving a linear heat transfer problem using GFM consists in to derive the Green's function. An auxiliary initial-boundary problem should be solved to determine the function. The solution to this problem can be obtained, for example, by using a method for the separation of variables, the Laplace transform or a method using images [4]. Alternative representations of Green's functions for two-dimensional heat conduction problems are presented by Melnikov [9], where the functions were obtained by means of a combination of the Laplace transform, the eigenfunction expansion method and the variation of parameters method. A set of Green's functions, useful in solving various heat conduction problems, is given in Ref. [4]. The form of the Green's function is of great significance because the above mentioned slow convergence of the solution can effect the accuracy of numerical calculations.

This paper presents an analytical solution to the heat conduction problem in a plate which is subjected to a moving heat source. The temperature of the plate changes because a limited area on the plate surface is heated by a heat source which moves along an elliptical trajectory. The temperature field in the rectangular plate is obtained as a solution to a three-dimensional heat conduction problem solved using the Green's function method.

2. Problem formulation

Consider a rectangular plate of uniform thickness h with edge lengths a and b, as shown in Fig. 1. The temperature $T(x,y,z,t)$ of the plate satisfies the differential equation of heat conduction:

$$\nabla_3^2 T - \frac{1}{\kappa}\frac{\partial T}{\partial t} + \frac{1}{\lambda} g(x,y,t) = 0 \tag{1}$$

where $\nabla_3^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, κ is thermal diffusivity, λ is thermal conductivity and $g(x,y,t)$ denotes a volumetric energy generation. In this study, it is assumed that the thermal energy is provided by a heat source which moves along a trajectory on the plate surface. Therefore, the function $g(x,y,z,t)$ takes the form

$$g(x,y,t) = \begin{cases} \frac{\Theta}{4\varepsilon^2}\delta(z-\frac{h}{2}) \\ \quad \text{for } \bar{x}(t)-\varepsilon < x < \bar{x}(t)+\varepsilon,\ \bar{y}(t)-\varepsilon < y < \bar{y}(t)+\varepsilon \\ 0 \quad \text{otherwise} \end{cases} \tag{2}$$

where Θ characterises the stream of heat, 2ε is the size of the quadratic element on the plate surface heated by the

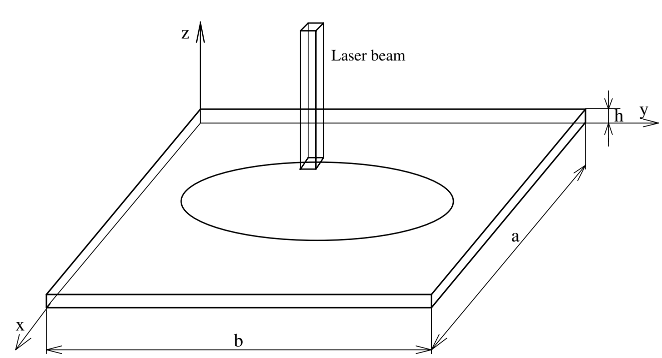

Fig. 1. Scheme of the considered plate.

heat source and $\delta()$ is the Dirac delta function. The functions $\bar{x}(t), \bar{y}(t)$, which describe the elliptical trajectory of the heat source are:

$$\bar{x}(t) = x_0 + A\cos\varphi t, \quad \bar{y}(t) = y_0 + B\sin\varphi t \tag{3}$$

where x_0, y_0, A, B are constants and φ is the angular velocity of the moving heat source. So that the heat source always remains within the plate area, it is assumed that the constants occurring in Eq. (3) satisfy the following inequalities: $A + \varepsilon < x_0 < a - A - \varepsilon$, $B + \varepsilon < y_0 < b - B - \varepsilon$.

The differential equation (1) is complemented by the following initial and boundary conditions:

$$T(x,y,z,0) = 0 \tag{4}$$

$$T(0,y,z,t) = T(a,y,z,t) = 0, \quad T(x,0,z,t) = T(x,b,z,t) = 0 \tag{5}$$

$$\lambda\partial T(x,y,h,t)/\partial z = \alpha_1[T_1 - T(x,y,h,t)] \tag{6}$$

$$\lambda\partial T(x,y,0,t)\partial z = -\alpha_0[T_0 - T(x,y,0,t)] \tag{7}$$

where α_0 and α_1 are the heat transfer coefficients, and T_0, T_1 are known temperatures of the surrounding medium.

3. Solution to the problem

The solution to the initial-boundary problem, which is given by Eq. (1) and conditions (4)–(7), is determined by using Green's function method. The Green's function $G(x,y,z,t,\xi,\eta,\varsigma,\tau)$ is a solution to the differential equation:

$$\left(\nabla_3^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right)G(x,y,z,t,\xi,\eta,\varsigma,\tau) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\xi)\delta(t-\tau) \tag{8}$$

and satisfies the homogeneous initial-boundary conditions, analogous to the initial and boundary conditions (4)–(7):

$$G|_{t=0} = 0, G|_{x=0} = G|_{x=a} = 0, G|_{y=0} = 0, G|_{y=b} = 0, \tag{9}$$

$$(\lambda\partial G/\partial z - \alpha_0 G)|_{z=0} = 0, \quad (\lambda\partial G/\partial z + \alpha_1 G)|_{z=h} = 0, \tag{10}$$

The Green's function for the considered heat conduction problem as a solution to the homogeneous differential problem (8)–(10) is presented in Appendix A. The application of a reciprocity relation [4]: $G(x,y,z,t,\xi,\eta,\zeta,\tau) = G(\xi,\eta,\zeta,-\tau,x,y,z,-t)$, in Eq. (8), yields

$$\left(\overline{\nabla}_3^2 + \frac{1}{\kappa}\frac{\partial T}{\partial\tau}\right)G(x,y,z,t,\xi,\eta,\zeta,\tau) = -\delta(x-\xi)\delta(y-\eta)\delta(z-\xi)\delta(t-\tau) \tag{11}$$

where $\overline{\nabla}_3^2 \equiv (\partial^2/\partial\xi^2) + (\partial^2/\partial\eta^2) + (\partial^2/\partial\zeta^2)$. Eq. (11) is then used to solve the problem.

The Green's function is applied to determine the temperature T in the plate. To this end the following steps should be performed:

- Replacement of variables x, y, z, t in Eq. (1) by ξ, η, ζ, τ, respectively.
- Multiplication of both sides of the equation obtained in the first step, by the Green's function $G(x,y,z,t,\xi,\eta,\varsigma,\tau)$.
- Integration of both sides of the equation obtained in the second step, with respect to ξ, η, ζ, τ in the intervals (0, a), (0, b), (0, h), (0, t), respectively.

As a result one obtains

$$\int_0^a\int_0^b\int_0^h\int_0^t\left[\left(\overline{\nabla}_3^2 - \frac{1}{\kappa}\frac{\partial T}{\partial\tau}\right)T(\xi,\eta,\varsigma,\tau) + \frac{1}{\lambda}g(\xi,\eta,\varsigma,\tau)\right] \times G(x,y,z,t,\xi,\eta,\zeta,\tau)\,d\zeta\,d\tau\,d\eta\,d\xi = 0 \tag{12}$$

Next, the integral in Eq. (12) is integrated by parts: the terms which include derivatives of the function T with respect to ξ, η, ζ, are integrated by parts twice and the term including the derivative with respect to τ is integrated once. After utilizing the initial and boundary conditions (4)–(7) and (9),(10), the following equation is obtained:

$$\int_0^a\int_0^b\int_0^h\int_0^t\left\{\left[\left(\overline{\nabla}_3^2 + \frac{1}{\kappa}\frac{\partial T}{\partial\tau}\right)G(x,y,z,t,\xi,\eta,\zeta,\tau)\right] \times T(\xi,\eta,\varsigma,\tau) + \frac{1}{\lambda}g(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau)\right\} \times d\tau\,d\zeta\,d\eta\,d\xi + \text{B.c.} = 0 \tag{13}$$

where

$$\text{B.c.} = \int_0^a\int_0^b\int_0^t[\mu_1 T_1 G|_{\zeta=h} + \mu_0 T_0 G|_{\zeta=0}]\,d\tau\,d\eta\,d\xi$$

Finally, using (11) in Eq. (13) and using the properties of the Dirac delta function, one obtains:

$$T(x,y,z,t) = \frac{1}{\lambda}\int_0^a\int_0^b\int_0^h\int_0^t g(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau) \times d\tau\,d\zeta\,d\eta\,d\xi + \text{B.c.} \tag{14}$$

Substituting the function $g(\xi,\eta,\zeta,\tau)$ given by (2) into Eq. (14), the temperature of the plate, $T(x,y,z,t)$, is expressed as

$$T(x,y,z,t) = \frac{\Theta}{4\varepsilon^2\lambda}\int_0^t\int_{\bar{x}(\tau)-\varepsilon}^{\bar{x}(\tau)+\varepsilon}\int_{\bar{y}(\tau)-\varepsilon}^{\bar{y}(\tau)-\varepsilon}G(x,y,z,t,\xi,\eta,h,\tau)\,d\eta\,d\xi\,d\tau + \text{B.c.} \tag{15}$$

The Green's function $G(x,y,z,t,\xi,\eta,\zeta,\tau)$ given by (A.1) is now utilized in Eq. (15). After evaluation of the integrals with respect to η, ξ, one obtains the temperature $T(x,y,z,t)$ in the form

$$T(x,y,z,t) = \frac{4\kappa\Theta}{\pi^2\varepsilon^2\lambda}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(z)\psi_n(h)}{jmQ_n}K_{jmn}(t) \times \sin\frac{jx\pi}{a}\sin\frac{my\pi}{b}\sin\frac{j\pi\varepsilon}{a}\sin\frac{m\pi\varepsilon}{b} + \text{B.c.} \tag{16}$$

where $\psi_n(z), Q_n$ are defined in Appendix A; and

$$K_{jmn}(t) = \int_0^t\sin\frac{j\pi\bar{x}(\tau)}{a}\sin\frac{m\pi\bar{y}(\tau)}{b}\exp(-\kappa\gamma_{jmn}^2(t-\tau))\,d\tau \tag{17}$$

with $\gamma_{jmn}^2 = \beta_n^2 + \left(\frac{j\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$, β_n are roots of equation (A.11).

The integral in Eq. (17), after taking into account the functions $\bar{x}(\tau)$ and $\bar{y}(\tau)$ which are given in Eq. (3), can be numerically calculated. An alternative way is to present the integrand in the form of an infinite series, and integrate the series "term by term". First, the function $K_{jmn}(t)$ is written as

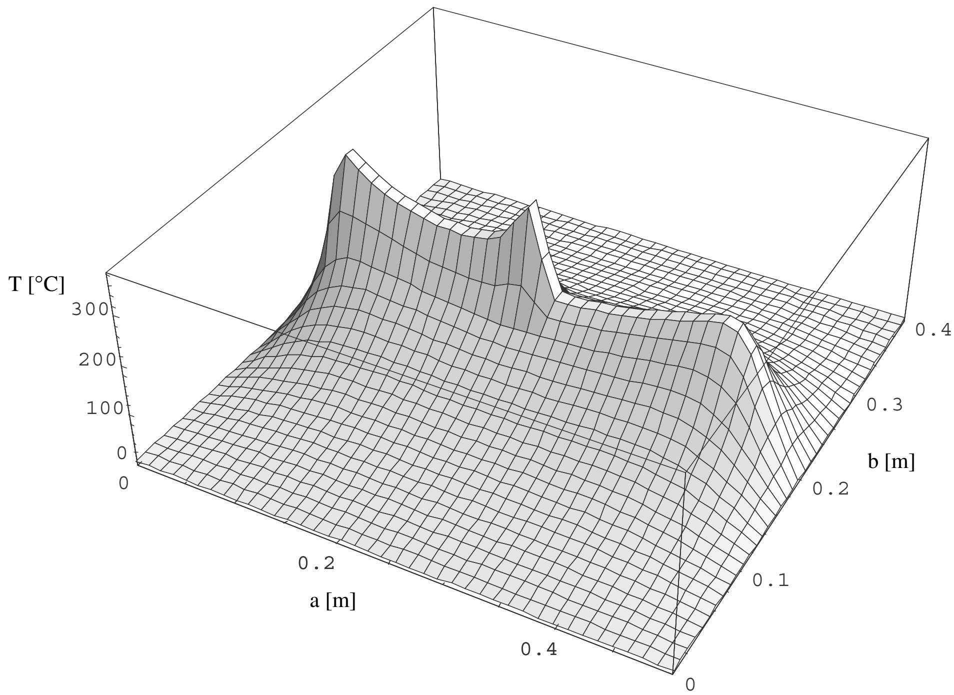

Fig. 2. Temperature distribution on the plate surface for $t = 3600$ s, when the heat source moves along a section parallel to an edge of the plate: $a = 0.5$ m, $b = 0.4$ m, $h = 0.02$ m, $A = 0.2$ m, $B = 0$.

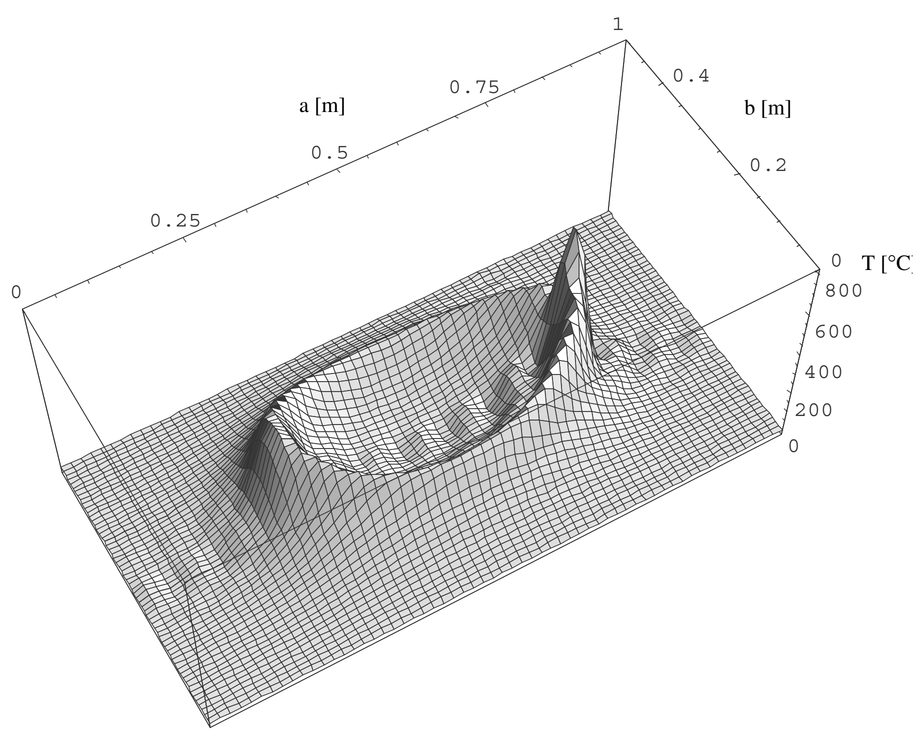

Fig. 3. Temperature distribution on the plate surface for $t = 3600$ s (heat source moves along an elliptical trajectory), $a = 1.0$ m, $b = 0.5$ m, $h = 0.01$ m, $A = 0.3$ m, $B = 0.2$ m.

$$
\begin{aligned}
K_{jmn}(t) = {} & \sin\frac{j\pi x_0}{a}\sin\frac{m\pi y_0}{b}\int_0^t \cos(\mu_j\cos\varphi\tau)\cos(v_m\sin\varphi\tau) \\
& \times \exp(-\kappa\gamma_{jmn}^2(t-\tau))\,\mathrm{d}\tau \\
& + \sin\frac{j\pi x_0}{a}\cos\frac{m\pi y_0}{b}\int_0^t \cos(\mu_j\cos\varphi\tau)\sin(v_m\sin\varphi\tau) \\
& \times \exp(-\kappa\gamma_{jmn}^2(t-\tau))\,\mathrm{d}\tau \\
& + \cos\frac{j\pi x_0}{a}\sin\frac{m\pi y_0}{b}\int_0^t \sin(\mu_j\cos\varphi\tau)\cos(v_m\sin\varphi\tau) \\
& \times \exp(-\kappa\gamma_{jmn}^2(t-\tau t))\,\mathrm{d}\tau \\
& + \cos\frac{j\pi x_0}{a}\cos\frac{m\pi y_0}{b}\int_0^t \sin(\mu_j\cos\varphi\tau)\sin(v_m\sin\varphi\tau) \\
& \times \exp(-\kappa\gamma_{jmn}^2(t-\tau))\,\mathrm{d}\tau
\end{aligned} \tag{18}
$$

where $\mu_j = \frac{j\pi A}{a}$ and $v_m = \frac{m\pi B}{b}$. The integrals occurring in Eq. (18) were evaluated and the results are presented in Appendix B.

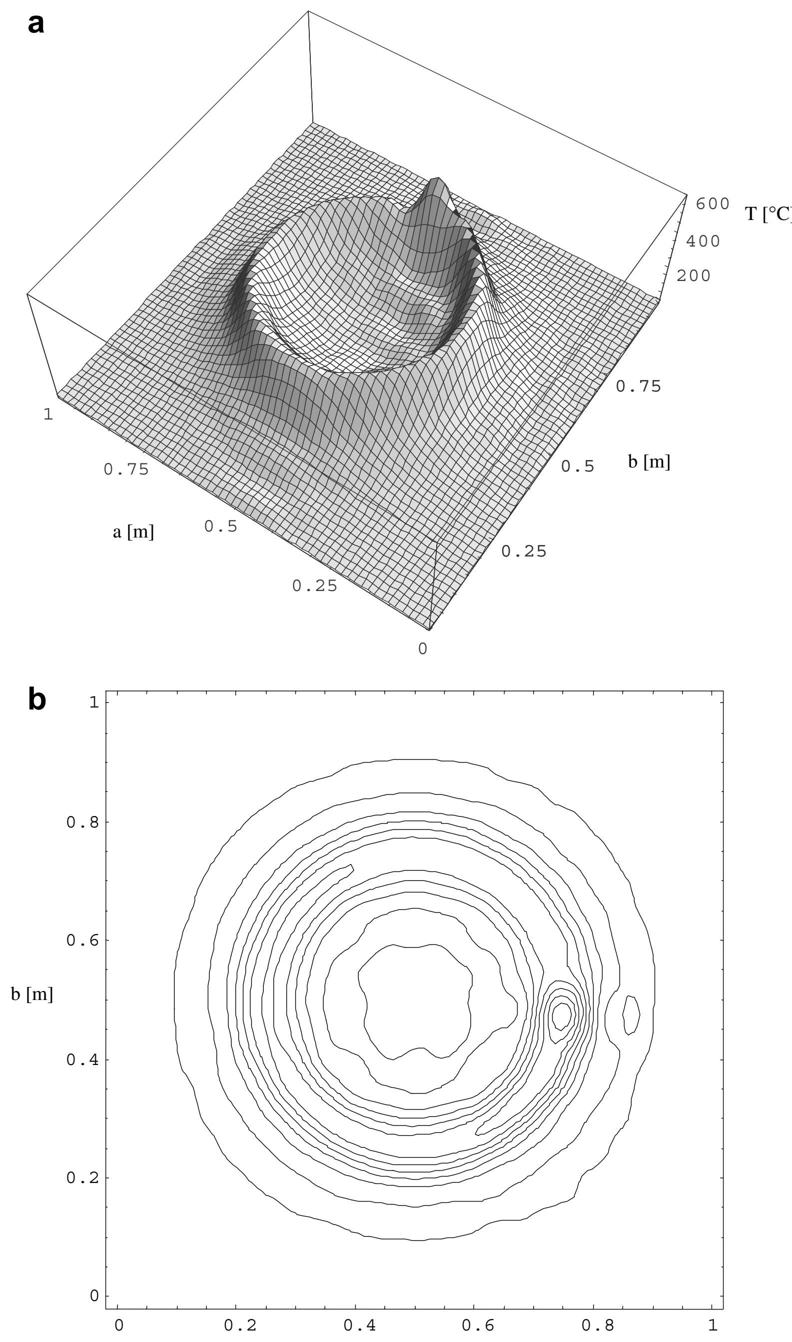

Fig. 4. Temperature distribution $T(x, y, z, t)$ on the square plate surface for $z = h$ at time $t = 3600$ s (heat source moves along an circular trajectory): (a) 3D plot, (b) isothermal lines; $a = 1.0$ m, $b = 1.0$ m, $h = 0.01$ m, $A = B = 0.25$ m.

4. Numerical examples

The solution to the considered three-dimensional heat conduction problem is used in the numerical investigation of the temperature distribution in a rectangular steel plate which is heated by a moving heat source. For each calculation, dimensional and physical properties of the plate and the heat source are: $\lambda = 51.4\ \mathrm{W\,m^{-1}\,K^{-1}}$, $\kappa = 1.29 \times 10^{-5}\ \mathrm{m^2\,s^{-1}}$, $\varphi = 0.2\,\pi\ \mathrm{rad\,s^{-1}}$, $\alpha_0 = \alpha_1 = 100\ \mathrm{W\,m^{-1}\,K^{-1}}$, $x_0 = a/2$, $y_0 = b/2$ and $\Theta = 10^5\ \mathrm{W\,m^{-2}}$. The temperature of the surrounding medium is assumed as $T_0 = T_1 = 100\ °\mathrm{C}$. The calculations were performed with use the *Mathematica* software.

In the first example, it was assumed that the heat source moves harmonically along a section parallel to one edge of the plate (it was assumed $B = 0$ in Eq. (3)). In this case, the calculations were performed for a plate with: $a = 0.5$ m, $b = 0.4$ m, $h = 0.02$ m and $A = 0.2$ m. The temperature distribution on the plate surface $z = h$, for time $t = 3600$–3610 s, is shown in Fig. 2. The time interval between observations of the temperature of the plate is set at 10 s because this is the time the heat source takes to execute a complete cycle of movement. A relative long time overall observation time (3600–3610 s) was selected in order that the process can be treated practically as in a steady state during this time. The location of the heat source is clearly visible in the figure and is shown by the temperature peak. The temperature is lower in front of heat source than behind the location of the heat source.

The temperature distribution on the plate surface when the heat source moves along an elliptical trajectory is illustrated in Fig. 3. The geometrical dimensions of the considered plate are: $a = 1.0$ m, $b = 0.5$ m, $h = 0.01$ m. The elliptical trajectory of the movement of the heat source is characterized by Eq. (3) with $A = 0.3$ m and $B = 0.2$ m. The temperature distribution on the plate surface is presented for $t = 3600$ s, when the heat source is located at the vertex of the ellipse. The temperature increases dramatically around the point of the plate at the moment of heat source transition, and the temperature changes over a low range only in the area of the middle of the ellipse.

In the third example, the temperature distribution on a square plate heated by a source which moves along a circular trajectory was determined. The centre of the circle coincides with the middle of the square. The following data were assumed for the calculations: $a = 1.0$ m, $b = 1.0$ m, $h = 0.01$ m, $A = B = 0.25$ m. The temperature distribution on the plate surface $z = h$, for $t = 3600$ s as a function of two variables x, y, is presented in Fig. 4(a), and the isothermal lines are shown in Fig. 4(b). Isotherms behind the heat source are much thicker than in the front and the position of the moving heat source can be clearly seen (Fig. 4(b)).

5. Conclusions

In this paper, an analytical model to describe the three-dimensional temperature field for a finite plate with a heat source which moves over its surface was established. The moving heat source causes cyclic heating of various plate areas. The temperature distribution in the considered plate in an analytical form was obtained using the time-dependent Green's function. The advantage of this approach is that a solution without any additional simplification can be obtained. Numerical calculation of the temperature distribution were performed using the analytical form of the solution. The changes in temperature on the plate surface are shown for a steel plate subjected to the activity of a moving heat source. The results are presented for the heat source moving along a section parallel to one edge of the plate, moving along an elliptical trajectory, and moving in a circular trajectory. The temperature is highest at the point of heat source location, but the temperature decreases behind the source. In the data assumed for the numerical calculations, in particular for the assumed velocity of the heat source, the temperature field of the plate area changes insignificantly over time apart from in a limited area around the heat source. Moreover, the temperature is considerably lower in the area around the centre of the circular or elliptical trajectory than near the trajectory.

Appendix A

The Green's function $G(x, y, z, t, \xi, \eta, \varsigma, \tau)$ of the three-dimensional heat conduction problem ((10) and (11)) can be expressed in a form of the product [6]

$$G(x, y, z, t, \xi, \eta, \varsigma, \tau) = \frac{4\kappa}{ab} H(t-\tau) G_X(x, \xi, t-\tau) G_Y(y, \eta, t-\tau) G_Z(z, \zeta, t-\tau) \tag{A.1}$$

where $H(t-\tau)$ is the Heaviside function, $G_X(x, \xi, t-\tau)$, $G_Y(y, \eta, t-\tau)$ and $G_Z(z, \zeta, \tau - t)$ are Green's functions of one-dimensional problems: in x-direction, y-direction and z-direction. The formulations of the considered one-dimensional initial- boundary problems are as follows:

x-direction:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) G_X(x, \xi, t-\tau) = -\delta(x-\xi)\delta(t-\tau) \tag{A.2}$$

$$G_X|_{t=0} = 0, G_X|_{x=0} = G_X|_{x=a} = 0 \tag{A.3}$$

y-direction:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) G_Y(y, \eta, t-\tau) = -\delta(y-\eta)\delta(t-\tau) \tag{A.4}$$

$$G_Y|_{t=0} = 0, G_Y|_{y=0} = G_Y|_{y=b} = 0 \tag{A.5}$$

z-direction:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{\kappa}\frac{\partial}{\partial t}\right) G_Z(z, \zeta, t-\tau) = -\delta(z-\zeta)\delta(t-\tau) \tag{A.6}$$

$$G_Z|_{t=0} = 0, (\lambda\partial G_Z/\partial z - \alpha_0 G_Z)|_{z=0} = 0, (\lambda\partial G_Z/\partial z + \alpha_1 G_Z)|_{z=h} = 0, \tag{A.7}$$

The Green's functions $G_X(x,\xi,t-\tau)$, $G_Y(y,\eta,t-\tau)$ and $G_Z(z,\zeta,\tau-t)$, are given by Beck et al. in Ref. [4]. The functions can be written in the form

$$G_X(x,\xi,t-\tau)=\frac{2}{a}\sum_{j=1}^{\infty}\exp\left(-\kappa\left(\frac{j\pi}{a}\right)^2(t-\tau)\right)\sin\frac{j\pi x}{a}\sin\frac{j\pi\xi}{a} \tag{A.8}$$

$$G_Y(y,\eta,t-\tau)=\frac{2}{b}\sum_{m=1}^{\infty}\exp\left(-\kappa\left(\frac{m\pi}{b}\right)^2(t-\tau)\right)\sin\frac{j\pi y}{b}\sin\frac{j\pi\eta}{b} \tag{A.9}$$

$$G_Z(z,\zeta,t-\tau)=\frac{2}{h}\sum_{n=1}^{\infty}\frac{1}{Q_n}\exp(-\kappa\beta_n^2(t-\tau))\psi_n(z)\psi_n(\zeta) \tag{A.10}$$

where $\psi_n(z)=\beta_n\cos\beta_n z+\mu_0\sin\beta_n z$, $Q_n=(\beta_n^2+\mu_0^2)\times\left(1+\frac{\beta_n^2+\mu_0\mu_1}{h\beta_n^2(\mu_0+\mu_1)}\sin^2\beta_n h\right)$, for $n=1,2,\ldots$, and β_n are roots of the equation:

$$\beta_n(\mu_0+\mu_1)\cos\beta_n h-(\beta_n^2-\mu_0\mu_1)\sin\beta_n h=0. \tag{A.11}$$

Appendix B

An analytic form of the integrals in equation (18) can be obtained by using the following relationships [10]:

$$\cos(r\sin u)=2\sum_{i=0}^{\infty}\chi_i J_{2i}(r)\cos 2iu \tag{B.1}$$

$$\sin(r\sin u)=2\sum_{i=0}^{\infty}J_{2i+1}(r)\sin(2i+1)u \tag{B.2}$$

and the relationships which can be obtain from (B.1, B.2):

$$\cos(r\cos u)=2\sum_{i=0}^{\infty}(-1)^i\chi_i J_{2i}(r)\cos 2iu \tag{B.3}$$

$$\sin(r\cos u)=2\sum_{i=0}^{\infty}(-1)^i J_{2i+1}(r)\cos((2i+1)u) \tag{B.4}$$

where $J_v(r)$ are Bessel functions and $\chi_0=0.5$, $\chi_i=1$ for $i=1,2,\ldots$. From Eqs. (B.1)–(B.4) one obtains

$$\cos(\mu_j\cos\varphi t)\cos(v_m\sin\varphi t)=4\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\chi_i\chi_k J_{2i}(\mu_j)J_{2k}(v_m)\cos 2i\varphi t\cos 2k\varphi t \tag{B.5}$$

$$\cos(\mu_j\cos\varphi t)\sin(v_m\sin\varphi t)=4\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\chi_i J_{2i}(\mu_j)J_{2k+1}(v_m)\cos 2i\varphi t\sin(2k+1)\varphi t \tag{B.6}$$

$$\sin(\mu_j\cos\varphi t)\cos(v_m\sin\varphi t)=4\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\chi_k J_{2i+1}(\mu_j)J_{2k}(v_m)\cos(2i+1)\varphi t\cos 2k\varphi t \tag{B.7}$$

$$\sin(\mu_j\cos\varphi t)\sin(v_m\sin\varphi t)=4\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i J_{2i+1}(\mu_j)J_{2k+1}(v_m)\cos(2i+1)\varphi t\sin(2k+1)\varphi t \tag{B.8}$$

Hence the function $K_{jmn}(t)$ in Eq. (18) may be then written as

$$\begin{aligned}K_{jmn}(t)=4\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^i\Big\{&\sin\frac{j\pi x_0}{a}\sin\frac{m\pi y_0}{b}\chi_i\chi_k J_{2i}(\mu_j)J_{2k}(v_m)I_{jmn}^{(1)}(t;2i,2k)\\&+\sin\frac{j\pi x_0}{a}\cos\frac{m\pi y_0}{b}\chi_i J_{2i}(\mu_j)J_{2k+1}(v_m)I_{jmn}^{(2)}(t;2i,2k+1)\\&+\cos\frac{j\pi x_0}{a}\sin\frac{m\pi y_0}{b}\chi_k J_{2i+1}(\mu_j)J_{2k}(v_m)I_{jmn}^{(1)}(t;2i+1,2k)\\&+\cos\frac{j\pi x_0}{a}\cos\frac{m\pi y_0}{b}J_{2i+1}(\mu_j)J_{2k+1}(v_m)I_{jmn}^{(2)}(t;2i+1,2k+1)\Big\}\end{aligned} \tag{B.9}$$

where

$$I_{jmn}^{(1)}(t;r,s)=\int_0^t\cos(r\varphi\tau)\cos(s\varphi\tau)\exp(-\kappa\gamma_{jmn}^2(t-\tau))\,d\tau \tag{B.10}$$

$$I_{jmn}^{(2)}(t;r,s)=\int_0^t\cos(r\varphi\tau)\sin(s\varphi\tau)\exp(-\kappa\gamma_{jmn}^2(t-\tau))\,d\tau \tag{B.11}$$

After evaluation the integrals (B.10), (B.11) are:

$$\begin{aligned}I_{jmn}^{(1)}(t;r,s)=&\frac{1}{C_{jmn}(r-s)C_{jmn}(r^*s)}\exp(-t\gamma_{jmn}^2\kappa)\gamma_{jmn}^2\kappa(\gamma_{jmn}^4\kappa^2+(r^2+s^2)\varphi^2)\\&+\frac{1}{2C_{jmn}(r-s)}[\gamma_{jmn}^2\kappa\cos[(r-s)t\varphi]+(r-s)\varphi\sin[(r-s)t\varphi]]\\&+\frac{1}{2C_{jmn}(r+s)}[\gamma_{jmn}^2\kappa\cos[(r+s)t\varphi]+(r+s)\varphi\sin[(r+s)t\varphi]]\end{aligned} \tag{B.12}$$

$$\begin{aligned}I_{jmn}^{(2)}(t;r,s)=&\frac{1}{C_{jmn}(r-s)C_{jmn}(r+s)}\exp(-t\gamma_{jmn}^2\kappa)s\varphi(\gamma_{jmn}^4\kappa^2+(-r^2+s^2)\varphi^2)\\&+\frac{1}{2C_{jmn}(r-s)}[(r-s)\varphi\cos[(r-s)t\varphi]-\gamma_{jmn}^2\kappa\sin[(r-s)t\varphi]]\\&+\frac{1}{2C_{jmn}(r+s)}[-(r+s)\varphi\cos[(r+s)t\varphi]+\gamma_{jmn}^2\kappa\sin[(r+s)t\varphi]]\end{aligned} \tag{B.13}$$

with $C_{jmn}(u)=\gamma_{jmn}^4\kappa^2+u^2\varphi^2$.

References

[1] G. Araya, G. Gutierrez, Analytical solution for a transient three-dimensional temperature distribution due to a moving laser beam, Int. J. Heat Mass Transfer 49 (2006) 4124–4131.

[2] P.J. Cheng, S.C. Lin, An analytical model for the temperature field in the laser forming of sheet metal, J. Mater. Process. Technol. 101 (2000) 260–267.

[3] O. Manca, B. Morrone, V. Naso, Quasi-steady three-dimensional temperature distribution induced by a moving circular Gaussian heat source in a finite depth solid, Int. J. Heat Mass Transfer 38 (7) (1995) 1305–1315.

[4] J.V. Beck, K.D. Cole, A. Haji-Sheikh, B. Litkouhi, Heat Conduction Using Green's Functions, Hemisphere, New York, 1992.

[5] J. Kidawa-Kukla, Application of the Green functions to the problem of the thermally induced vibration of a beam, J. Sound Vibr. 262 (2003) 865–876.

[6] J. Kidawa-Kukla, Analysis of beam vibration induced by moving heat source, in: H.A. Mang, F.G. Rammerstorfer, J. Eberhardsteiner, J. (Eds.), Proceedings of the Fifth World Congress on Computational

Mechanics (WCCM V), July 7–12, 2002, Vienna, Austria. Vienna University of Technology, Austria, ISBN 3-9501554-0-6, <http://wccm.tuwien.ac.at>.

[7] J. Kidawa-Kukla, Temperature distribution in a rectangular plate subjected to the activity of a mobile heat source, in: XII International Science and Engineering Conference. Machine-Building and Technosphere of the XXI Century. 12–17 September 2005, Sevastopol, vol. 5, 2005, pp. 111–115.

[8] J.V. Beck, A. Haji-Sheikh, D.E. Amos, D. Yen, Verification solution for partial heating of rectangular solids, Int. J. Heat Mass Transfer 47 (2004) 4243–4255.

[9] Y.A. Melnikov, An alternative construction of Green's functions for the two-dimensional heat equation, Eng. Anal. Bound. Elem. 24 (2000) 467–475.

[10] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solids, Clarendon Press, London, 1959.